JOURNAL OF COMBINATORIAL THEORY, Series A **71**, 168–172 (1995)

Note

On Subsets of Finite Abelian Groups with No 3-Term Arithmetic Progressions

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Communicated by the Managing Editors

Received September 3, 1993

Let G be a finite abelian group of odd order and let $D(G)$ denote the maximal cardinality of a subset $A \subset G$ which does not contain a 3-term arithmetic progression. It is shown that $D(\mathbf{Z}_{k_1} \oplus \cdots \oplus \mathbf{Z}_{k_n}) \leqslant 2((k_1 \cdots k_n)/n)$. Together with results of Szemerédi and Heath-Brown it implies that there exists a $\beta > 0$ such that $D(G) = O(|G|/(\log |G|)^\beta)$ for all G.

INTRODUCTION

Let G be a finite abelian group of odd order and let $D(G)$ denote the maximal cardinality of a subset $A \subset G$ which does not contain a 3-term arithmetic progression.

In a fundamental paper [6], Roth proved that $D(\mathbf{Z}_m) = O(m/\log\log m)$. Roth's estimate was recently improved by Szemerédi [7] and Heath-Brown [5]:

THEOREM 1.1 (Szemerédi, Heath-Brown). *$D(\mathbf{Z}_m) = O(m/(\log m)^\alpha)$ for some fixed $\alpha > 0$.*

Brown and Buhler [3] and Frankl, Graham and Rödl [4] proved that $D(G) = o(|G|)$ for all G (here, and throughout the paper G denotes a finite abelian group of odd order).

In this note we are interested in $D(G)$ for groups with many constituents. Alon and Dubiner [1, 2] asked whether there exists a constant $c < 3$ such that $D(\mathbf{Z}_3^n) \leqslant c^n$. I. Ruzsa has recently proved that $D(\mathbf{Z}_3^n) = O(3^n/\sqrt{n})$ (private communication from N. Alon).

0097-3165/95 \$12.00

Let $\mathbf{Z}_{k_1} \oplus \cdots \oplus \mathbf{Z}_{k_n}$ be the canonical decomposition of G where $k_1 | \cdots | k_n$, and let $c(G) = n$ denote the number of constituents of G.

Here we use Roth's approach [6] to show:

THEOREM 1.2. $D(G) \leqslant 2\,|G|/c(G)$.

Combining this with Theorem 1.1 we obtain

COROLLARY 1.3. *There exists a $\beta > 0$ such that $D(G) = O(|G|/(\log |G|)^\beta)$ for all G.*

We conclude the introduction with several properties of the Fourier transform on a finite abelian group G:

Let $\hat{G}$ denote the character group of G.

The *Fourier transform* of a function $f\colon G \to \mathbf{C}$, is the function $\hat{f}\colon G \to \mathbf{C}$ defined by $\hat{f}(\chi) = \sum_{x \in G} f(x)\,\chi(-x)$.

G is naturally isomorphic to $\hat{\hat{G}}$ (via $x \to \varphi_x$ where $\varphi_x(\chi) = \chi(x)$) and $\hat{\hat{f}}(x) = |G|\, f(-x)$.

The Fourier transform satisfies the *Parseval identity* $\sum_{x \in \hat{G}} |\hat{f}(\chi)|^2 = |G| \sum_{x \in G} |f(x)|^2$.

The *convolution* of two functions f and g is given by $f * g(x) = \sum_{y \in G} f(y)\, g(x-y)$, and satisfies $\widehat{f * g}(\chi) = \hat{f}(\chi)\, \hat{g}(\chi)$. The k-fold convolution $f * \cdots * f$ is denoted by f^{*k}.

The characteristic function 1_S of a subset $S \subset G$ is given by $1_S(x) = 1$ if $x \in S$ and zero otherwise.

If H is a subgroup of G then $\widehat{1_H}(\chi) = |H|\, 1_{H^\perp}(\chi)$ where $H^\perp = \{\chi \in \hat{G}\colon \chi(x) = 1 \textit{ for all } x \in H\}$.

Let χ_0 denote the trivial character and let $\delta(\chi) = 1$ if $\chi = \chi_0$ and zero otherwise. If $f(x) = 1$ for all $x \in G$ then $\hat{f}(\chi) = |G|\, \delta(\chi)$.

If $\varphi\colon G \to H$ is a homomorphism, then $c(\ker \varphi) \geqslant c(G) - c(H)$. In particular $c(\ker \chi) \geqslant c(G) - 1$ for all $\chi \in \hat{G}$.

ON SUBSETS OF $\mathbf{Z}_{k_1} \oplus \cdots \oplus \mathbf{Z}_{k_n}$ WITH NO 3-TERM ARITHMETIC PROGRESSIONS

Proof of Theorem 1.2. Let $d(n)$ denote the supremum of $D(G)/|G|$ as G ranges over all G's such that $c(G) \geqslant n$.

We shall show by induction that $d(n) \leqslant 2/n$.

Let G be a group with $c(G) \geqslant n \geqslant 2$ and let A be a subset of G which does not contain a 3-term arithmetic progression.

Let $B = -2A = \{-2a: a \in A\}$. Let $f(\chi) = \widehat{1_A}(\chi)$, $g(\chi) = \widehat{1_B}(\chi)$, and $h(\chi) = d(n-1)\,|G|\,\delta(\chi)$.

Since A does not contain a 3-term arithmetic progression it follows that $1_A^{*2} * 1_B(0) = |A|$, which implies

$$\sum_{\chi \in \hat{G}} f(\chi)^2 g(\chi) = \sum_{\chi \in \hat{G}} \widehat{1_A^{*2} * 1_B}(\chi) = |G|\, 1_A^{*2} * 1_B(0) = |G|\,|A|. \tag{1}$$

On the other hand

$$\sum_{\chi \in \hat{G}} f(\chi)^2 h(\chi) = f(\chi_0)^2 h(\chi_0) = |G|\,|A|^2 d(n-1). \tag{2}$$

Let $u(x) = d(n-1) - 1_B(x)$, then $\hat{u}(\chi) = h(\chi) - g(\chi)$.

Proposition 2.1.

$$\max_{\chi \in \hat{G}} |\hat{u}(\chi)| = d(n-1)\,|G| - |A|.$$

Proof. Let $\chi \in \hat{G}$ and consider the subgroup $W = \ker \chi$. For each $x \in G$, $W \cap x - B$ does not contain a 3-term arithmetic progression, therefore $|W \cap x - B| \leqslant |W|\, d(c(W)) \leqslant |W|\, d(n-1)$. It follows that

$$1_W * u(x) = \sum_{w \in W} u(x-w) = |W|\, d(n-1) - |B \cap x - W| \geqslant 0.$$

This holds as $|B \cap x - W| = |W \cap x - B|$.

Applying the Fourier transform we obtain

$$\begin{aligned} |W|\,|\hat{u}(\chi)| = |\widehat{1_W * u}(\chi)| &= \left| \sum_{x \in G} 1_W * u(x)\, \chi(-x) \right| \leqslant \sum_{x \in G} 1_W * u(x) \\ &= |W|\,|G|\, d(n-1) - \sum_{x \in G} |B \cap x - W| \\ &= |W|\,|G|\, d(n-1) - |B|\,|W|. \end{aligned}$$

It follows that $|\hat{u}(\chi)| \leqslant d(n-1)\,|G| - |B| = d(n-1)\,|G| - |A|$.

On the other hand

$$\hat{u}(\chi_0) = h(\chi_0) - g(\chi_0) = d(n-1)\,|G| - |A|$$

which completes the proof of the proposition. ∎

PROPOSITION 2.2.

$$|d(n-1)\,|A|-1| \leqslant d(n-1)\,|G|-|A|.$$

Proof. Combining (1), (2), Proposition 2.1 and the Parseval identity we obtain

$$\begin{aligned}|G|\,|A|\,|d(n-1)\,|A|-1| &= \left|\sum_{\chi\in\hat{G}} f(\chi)^2\,(h(\chi)-g(\chi))\right| \\ &\leqslant \sum_{\chi\in\hat{G}} |f(\chi)|^2 \max_{\chi\in\hat{G}} |\hat{u}(\chi)| \\ &= |G|\,|A|\,(d(n-1)\,|G|-|A|). \quad \blacksquare\end{aligned}$$

Proposition 2.2 together with the induction hypothesis imply that

$$\frac{|A|}{|G|} \leqslant \frac{|G|^{-1}+d(n-1)}{1+d(n-1)} \leqslant \frac{3^{-n}+d(n-1)}{1+d(n-1)} \leqslant \frac{3^{-n}+(2/n-1)}{1+(2/n-1)} \leqslant \frac{2}{n}. \quad \blacksquare$$

Proof of Corollary 1.3. By Theorem 1.1 $D(\mathbf{Z}_m)=O(m/(\log m)^{\alpha})$ for some fixed $\alpha>0$.

Let $\beta=\alpha/1+\alpha$ and consider the following two cases:

If $c(G)>(\log |G|)^{\beta}$ then we are done by Theorem 1.2.

Otherwise $t=c(G)\leqslant(\log |G|)^{\beta}$. Let A be a subset of G which does not contain a 3-term arithmetic progression, and let H be a *cyclic* subgroup of G such that $|H|\geqslant |G|^{1/t}$. By averaging there exists a coset $x+H$ such that

$$|A\cap x+H| \geqslant \frac{|A|\,|H|}{|G|}.$$

By Theorem 1.1 $|A\cap x+H|=O(|H|/(\log |H|)^{\alpha})$ therefore

$$|A|=O\left(\frac{|G|}{(\log |H|)^{\alpha}}\right)=O\left(\frac{|G|\,t^{\alpha}}{(\log |G|)^{\alpha}}\right)=O\left(\frac{|G|}{(\log |G|)^{\beta}}\right). \quad \blacksquare$$

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